Competitive Equilibrium Models

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Contents

1 The Econometrics of Price Discovery

1.1 Madhavan, Richardson and Roomans(1997)

For the trade direction indicator q_t in the Roll model, Madhavan, Richardson, and Roomans (1997) allow for serial dependence. Suppose that $q_t \in \{-1, +1\}$, and that:

$$\Pr(q_{t+1} = +1 \mid q_t = +1) = \Pr(q_{t+1} = -1 \mid q_t = -1) = \alpha,$$

$$\Pr(q_{t+1} = +1 \mid q_t = -1) = \Pr(q_{t+1} = -1 \mid q_t = +1) = 1 - \alpha.$$

The parameter α is called the continuation probability. If $\alpha = \frac{1}{2}$, trade directions are uncorrelated. If $\frac{1}{2} < \alpha < 1$, trade directions are *persistent* (buys tend to follow buys, etc.) With this structure, q_t may be expressed as an AR(1) process $q_t = \phi q_{t-1} + v_t$, where $\mathbb{E}[v_t] = 0$, $\mathbb{E}[v_t^2] = \sigma_v^2$, and $\mathbb{E}[v_t v_{t-k}] = 0$ for $k \neq 0$. The model may be analyzed by constructing a table of the eight possible realizations (paths) of (q_t, q_{t+1}, q_{t+2}).

(a) Assuming that q_t is equally likely to be ± 1, compute the probabilities of each path. Show that $\phi = 2\alpha - 1$.

(b) Compute v_{t+1} and v_{t+2}. Verify that $\mathbb{E}[v_{t+1}] = \mathbb{E}[v_{t+2}] = 0$ and show that $\mathrm{Cov}(v_{t+1}, v_{t+2}) = \mathbb{E}[v_{t+1} v_{t+2}] = 0$.

(c) Demonstrate that the v_t values are not serially independent by verifying that $\mathrm{Cov}(v_{t+1}, v_{t+2}^2) \neq 0$.

Solution

Let's first make a table for the joint probability density Assuming $q_t \in \{-1, +1\}$ with equal probability and Markov transition probability α, the joint probabilities for all 8 paths are as follows

Path	(q_t, q_{t+1}, q_{t+2})	Probability Expression	Probability Value
1	$(+1, +1, +1)$	$0.5 \cdot \alpha \cdot \alpha$	$\frac{1}{2}\alpha^2$
2	$(+1, +1, -1)$	$0.5 \cdot \alpha \cdot (1-\alpha)$	$\frac{1}{2}\alpha(1-\alpha)$
3	$(+1, -1, +1)$	$0.5 \cdot (1-\alpha) \cdot (1-\alpha)$	$\frac{1}{2}(1-\alpha)^2$
4	$(+1, -1, -1)$	$0.5 \cdot (1-\alpha) \cdot \alpha$	$\frac{1}{2}\alpha(1-\alpha)$
5	$(-1, -1, -1)$	$0.5 \cdot \alpha \cdot \alpha$	$\frac{1}{2}\alpha^2$
6	$(-1, -1, +1)$	$0.5 \cdot \alpha \cdot (1-\alpha)$	$\frac{1}{2}\alpha(1-\alpha)$
7	$(-1, +1, -1)$	$0.5 \cdot (1-\alpha) \cdot (1-\alpha)$	$\frac{1}{2}(1-\alpha)^2$
8	$(-1, +1, +1)$	$0.5 \cdot (1-\alpha) \cdot \alpha$	$\frac{1}{2}\alpha(1-\alpha)$

Now, we need to Identify the expected value and variance of the variables. these may be given by the joint probability distribution. Take for example the case where $q_t = q_{t+1} = q_{t+2} = +1$ we may find the joint probability of the variable. Now, one can easily show by recursive backward induction the property of AR models of the form $q_t = \theta q_{t-1} + v_t$ by allowing for recursive substitution of the term for q_k, hence getting the form

$$q_{t+n} = \sum_{j=1}^{n} (\theta)^j v_j + \alpha$$

Hence, we can obtain all the innovations for each time $t \in \{0, \ldots, t\}$. Specially in our case, since all covariances of the order 2 and above are zero, the model breaks down to

$$v_{t+2} = q_{t+2} - \phi q_{t+1}$$

$$v_{t+1} = q_{t+1} - \phi q_t$$

Also, from the origianal equation, $q_t = \phi q_{t-1} + v_t$ we have the following relation - $\phi = \frac{Cov(q_{t-1}, q_t)}{Var(q_t)}$. It may be conveninent to show that the variance of the latter is 1. Hence, substituting $t = t+1$, we have $\phi = Cov(q_t, q_{t+1}) = E(q_t q_{t+1}) = q\alpha - 1$. Now the only thing left is to demonstrate that v_t is not serially independent. This we shall tackle soon.

Now, consider the condition for independence - it is not the same as that of uncorrelatedness. The mathematical condition for independence is given as

$$Cov(x, f(y)) \neq 0, \forall f$$

As a general procedure, we pick the function $f(x) = x^2$ to prove this. Hence, it may be justified to show why $Cov(v_{t+1}, v_{t+2}^2) \neq 0$, which on calculation may be found to have dependence on higher order non-zero moments.

1.2 Serial Autocorrelation in Trade Direction

The Roll model assumes that trade directions are serially uncorrelated i.e. $\text{Corr}(q_t, q_s) = 0$ for $t \neq s$. In practice, one often finds positive autocorrelation (see Hasbrouck and Ho (1987), Choi, Salandro, and Shastri (1988)). Suppose that $\text{Corr}(q_t, q_{t-1}) = \rho > 0$ and $\text{Corr}(q_t, q_{t-k}) = 0$ for $k > 1$. Suppose that ρ is known.

(a) Show that

$$\text{Var}(\Delta p_t) = 2c^2(1-\rho) + \sigma_u^2, \quad \text{Cov}(\Delta p_t, \Delta p_{t-1}) = -c^2(1-2\rho),$$

$$\text{Cov}(\Delta p_t, \Delta p_{t-2}) = -c^2\rho, \quad \text{and} \quad \text{Cov}(\Delta p_t, \Delta p_{t-k}) = 0 \quad \text{for} \quad k > 2.$$

(b) Suppose that $0 < \rho < 1$ describes the true structural model. We compute an estimate of c, denoted $\hat{c}$, assuming that the original Roll model is correct. Show that $\hat{c} < c$, that is, that $\hat{c}$ is biased downward.

Solution

Consider the following martingale processes followed by the fundamental price - $m_t = m_{t-1} + u_t$ and the process followed by the actual price given by $p_t = m_t + c_t q_t$. Hence, by following the differences, we get $\Delta p_t = u_t + c - t(\Delta q_t)$. Now all we need to do is follow the covariance structures for the solution.

$$Var(\Delta p_t) = Var(u_t) + c_t^2 Var(q_t - q_{t-1}) + 2Cov(u_t, \Delta q_t)$$

$$Var(\Delta p_t) = \sigma_u^2 + 2c^2(1-\rho) \tag{1}$$

This is because we need to remember the fact that there's a covariance terms that hasn't been included before that's now present because of the correlation. Now, heading on the autocovariances, we find the following

$$Cov(\Delta p_t, \Delta p_{t-1}) = Cov(p_t - p_{t-1}, p_{t-1}, p_{t-2}) \tag{2}$$

expanding and solving we can use the distributive properties of covariances given as

$$Cov(a-b, c-d) = Cov(a,c) - Cov(a,d) - Cov(b,c) + Cov(b,d)$$

this is a very important implication in our expansion. Let's use this to solve our problem. Hence, (2) may be given as

$$Cov(p_t, p_{t-1}) - Cov(p_t, p_{t-2}) - Var(p_{t-1}) - Cov(p_{t-1}, p_{t-2}) \tag{3}$$

Since we also have $Var(q_t) = Var(q_{t-1}) = 1$ and $Cov(q_t, q_{t-1}) = Cov(q_{t-1}, q_{t-2}) = \rho$ as well as $Cov(q_t, q_{t-2}) = 0$ by the autoregressive property of the model. Hence, on a simple expansion, we have

$$Cov(\Delta p_t, \Delta p_{t-1}) = -c^2(1-2\rho) \tag{4}$$

Now, let's estimate a measure of the Roll model given as $\hat{c}$. this measurement assumes that the original model is true. Hence, in case of a non-zero correlation, we have $\frac{\hat{c}^2}{2} = c^2(1-2\rho)$ giving us $\hat{c} = c \times \sqrt{1-2\rho}$ which is essentially $< c$ for all $0 < \rho < 1$

1.3 Roll Models with Correlated Fundamentals

The basic Roll model assumes that trade directions are uncorrelated with changes in the efficient price: $\text{Corr}(q_t, u_t) = 0$. Suppose that $\text{Corr}(q_t, u_t) = \rho$, where ρ is known and $0 < \rho < 1$. The idea here is that a buy order is associated with an increase in the security value, a connection that will be developed in models of asymmetric information. Suppose that ρ is known.

(a) Show that

$$\begin{aligned}\mathrm{Var}(\Delta p_t) &= 2c^2 + \sigma_u^2 + 2c\rho\sigma_u,\\ \mathrm{Cov}(\Delta p_t, \Delta p_{t-1}) &= -c(c + \rho\sigma_u),\\ \mathrm{Cov}(\Delta p_t, \Delta p_{t-k}) &= 0 \quad \text{for } k > 1.\end{aligned}$$

(b) Suppose that $0 < \rho < 1$ describes the true structural model. We compute an estimate of c, denoted $\hat{c}$, assuming that the original Roll model is correct. Show that $\hat{c} > c$, that is, that $\hat{c}$ is biased upward.

Solution
Assume that there's correlation between order flow and fundamental value i.e. a buy today triggers a higher fundamental value tomorrow, then we can rework the previous question in the following manner given $Cov(q_t, u_t) \neq 0$.

$$Var(\Delta p_t) = \sigma_u^2 + 2Cov(u_t, q_t - q_{t-1}) \tag{5}$$

Since we have $c = -\sqrt{Cov(\Delta p_t, \Delta p_{t-1})}$ then we can assume that $Cov(u_t, q_{t-1}) = 0$ and conclude that

$$\mathrm{Var}(\Delta p_t) = \sigma_u^2 + 2c^2 + 2c\rho\sigma_u$$

this is because we have a memory of only the current trade and not the previous trade directions. For us, the previous trade directions q_{t-1} and beyond are immaterial. In a similar fashion, it may also be possible to show that $Cov(\Delta p_t, \Delta p_{t-1}) = -c(c + \rho\sigma_u$.

1.4 Price History means Innovation History

Consider the standard generalized version of the Roll(1984) model where the price evolves according to $p_t = m_t + cq_t$, fundamental value evolves according to the process $m_t = m_{t-1} + \lambda q_t + u_t$. If the first difference in the price process $\Delta p_t = p_t - p_{t-1}$ is observed, then (a) show that p_t my be realized as the conditional expectation of its past processes, i.e. $\mathbb{E}(p_t|p_{t-1}, p_{t-2}, \dots)$ (b) the fact that the price change process Δp_t may be shown as an AR(1) process and (c) that price history is the same as innovation history, or in other words that Δp_t can also be represented as a MA(1) process.

Solution

Since q_t is independent of u_t, we can safely conclude that prices are basically conditional expectations of their past values. It may first be convenient to show that this price follows an AR(1) process, then we can try and invert the same.

Consider the first order difference in prices Δp_t which is

$$\Delta p_t = (\lambda + c)q_t - cq_{t-1} + u_t$$

and

$$\Delta p_{t-1} = (\lambda + c)q_{t-1} + cq_{t-2} + u_{t-1}$$

Now, it may be convenient to use the Yule-Walker projection of Δp_t on its own moments given by

$$Var(\Delta p_t) = (\lambda + c)^2 + c^2 + \sigma_u^2$$

$$Cov(\Delta p_t, \Delta p_{t-1}) = -c(\lambda + c)$$

In an ordinary linear projection, we can define the dependent variable and the regressor by the following relation

$$y = \frac{Cov(y, x_1)}{Var(x_1)}x_1 + \frac{Cov(y, x_2)}{Var(x_2)}x_2 + \cdots + \frac{Cov(y, x_n)}{Var(x_n)}x_n$$

Hence, we have the first order autocorrelation given as

$$\beta_1 = \frac{Cov(\Delta p_t, \Delta p_{t-1})}{Var(\Delta p_{t-1})}$$

since $Var(\Delta p_t) = Var(\Delta p_{t-1})$, we can conclude that

$$\beta_1 = -\frac{c(\lambda + c}{(\lambda + c)^2 + c^2 + \sigma_u^2}$$

if $c > 0$, it may be possible to show that the best predictor is an AR(1) process and that $|\beta_1| < 1$. Hence, we can conclude that such a process is a very good predictor since all auto-covariances of the order 2 and above are zero.
Now, all we need to do is to invert this AR(1) process to a moving average kind of process. To do this, consider the AR(1) process

$$\Delta p_t = \beta_1 \Delta p_{t-1} + \varepsilon_t, \quad \text{where } |\beta_1| < 1, \text{ and } \varepsilon_t \sim \text{i.i.d. } (0, \sigma^2)$$

We now recursively substitute lagged values of Δp_{t-j} using the AR(1) equation

$$\begin{aligned}
\Delta p_t &= \beta_1 \Delta p_{t-1} + \varepsilon_t \\
&= \beta_1(\beta_1 \Delta p_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \beta_1^2 \Delta p_{t-2} + \beta_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \beta_1^3 \Delta p_{t-3} + \beta_1^2 \varepsilon_{t-2} + \beta_1 \varepsilon_{t-1} + \varepsilon_t \\
&\ \ \vdots \\
&= \sum_{j=0}^{\infty} \beta_1^j \varepsilon_{t-j}
\end{aligned}$$

Hence, the AR(1) process can be represented as a Moving Average of infinite order, given by

$$\Delta p_t = \sum_{j=0}^{\infty} \beta_1^j \varepsilon_{t-j}$$

This expansion is valid as long as the stationarity condition $|\beta_1| < 1$ is satisfied, which ensures geometric decay and convergence. Hence, price history is equivalent to innovation history.

1.5 Variance of the Filtering Error

Consider a generalized Roll model where $p_t = m_t + cq_t$, we define the tracking error as $\sigma_s^2 = Var(p_t - m_t)$. Show that (a) this variance has a finite lower bound which may be achieved only under the circumstance that $\sigma_u^2 = 0$ or more specifically, $u_t = 0, \forall t$ and (b) calculate the value of this tracking error in terms of c, λ, σ_u^2 and show that under the first order condition, this particular expression is lower bounded with a finite value.

Solution

Generally speaking, we have the following relation given as $s_t = p_t - m_t$ i.e. adding and subtracting the term f_t, we have $s_t = (p_t - f_t) + (m_t - f_t)$ where $f_t = \mathbb{E}(m_t | p_t, p_{t-1})$. Hence, it may be possible to calculate the variance of this first term.

$$\sigma_s^2 = Var(p_t - f_t) + Var(m_t - f_t) \tag{6}$$

since $f_t = p_t + \theta\epsilon_t$ from our famous condition "Price history is innovation history", we may be able to show that $Var(f_t - p_t) = \theta^2\sigma_\epsilon^2$ or the whole term for Eq. (6) has a minimum value. Now, we must proceed to calculate the value of the second term in Eq. (6). We are also given that

$$\theta = -\frac{c(c+\lambda)}{\sigma_\epsilon^2} \Rightarrow \theta^2 = \frac{c^2(c+\lambda)^2}{\sigma_\epsilon^4}$$

Substituting back in our equation we have

$$\text{Var}(m_t - f_t) = \frac{c^2(c+\lambda)^2}{\sigma_\epsilon^4} \cdot \sigma_\epsilon^2 = \frac{c^2(c+\lambda)^2}{\sigma_\epsilon^2}$$

and since

$$\sigma_\epsilon^2 = \frac{A}{1+\theta}, \quad \text{where } A = c^2 + (c+\lambda)^2 + \sigma_u^2 \Rightarrow \text{Var}(m_t - f_t) = \frac{c^2(c+\lambda)^2(1+\theta)}{A}$$

$$\sigma_s^2 = \frac{1}{2}\left[c^2 + (c+\lambda)^2 + \sigma_u^2 - \frac{(\lambda^2 + \sigma_u^2)(2c+\lambda)^2}{(2c+\lambda)^2 + \sigma_u^2}\right] \tag{7}$$

1.6 Lagged Delay Dynamics

The beliefs of market participants at time t are summarized in m_t, where $m_t = m_{t-1} + w_t$, with $w_t \sim \text{iid}(0, \sigma_w^2)$. But due to operational delays, trades actually occur relative to a lagged value

$$p_t = m_{t-1} + cq_t,$$

where $q_t \in \{-1, +1\}$ represents the direction of the trade and is assumed to be iid with zero mean and unit variance. Find out (a) What are the autocovariances of the price process p_t? and (b) the moving average representation of p_t?

Solution

We compute

$$\begin{aligned}\mathrm{Cov}(\Delta p_t, \Delta p_{t-1}) &= \mathrm{Cov}\left(w_{t-1} + c(q_t - q_{t-1}),\ w_{t-2} + c(q_{t-1} - q_{t-2})\right) \\ &= \underbrace{\mathrm{Cov}(w_{t-1}, w_{t-2})}_{=0} + \underbrace{c\,\mathrm{Cov}(w_{t-1}, q_{t-1} - q_{t-2})}_{=0} \\ &\quad + \underbrace{c\,\mathrm{Cov}(q_t - q_{t-1}, w_{t-2})}_{=0} + c^2\,\mathrm{Cov}(q_t - q_{t-1},\ q_{t-1} - q_{t-2})\end{aligned} \tag{8}$$

Now compute the final term

$$\mathrm{Cov}(q_t - q_{t-1}, q_{t-1} - q_{t-2}) = \mathbb{E}[(q_t - q_{t-1})(q_{t-1} - q_{t-2})]$$

$$= \mathbb{E}[q_t q_{t-1} - q_t q_{t-2} - q_{t-1}^2 + q_{t-1} q_{t-2}]$$

Using independence and zero mean

$$\mathbb{E}[q_t q_{t-1}] = \mathbb{E}[q_t q_{t-2}] = \mathbb{E}[q_{t-1} q_{t-2}] = 0, \quad \mathbb{E}[q_{t-1}^2] = 1$$

$$\Rightarrow \mathrm{Cov}(q_t - q_{t-1}, q_{t-1} - q_{t-2}) = -1$$

Therefore

$$\mathrm{Cov}(\Delta p_t, \Delta p_{t-1}) = -c^2 \tag{9}$$

Now, we proceed to calculate the Variance of the price change given as

$$\mathrm{Var}(\Delta p_t) = \mathrm{Var}(w_{t-1}) + c^2 \mathrm{Var}(q_t - q_{t-1})$$

Since q_t, q_{t-1} are iid with unit variance:

$$\text{Var}(q_t - q_{t-1}) = \text{Var}(q_t) + \text{Var}(q_{t-1}) = 2$$

$$\gamma_0 = \text{Var}(\Delta p_t) = \sigma_w^2 + 2c^2 \tag{10}$$

1.7 Correlated Lagged Delays

Delays may also lead to price adjustments that do not instantaneously correct. Suppose the fundamental value evolves as:

$$m_t = m_{t-1} + w_t,$$

but the observed price adjusts only partially toward the fundamental value

$$p_t = p_{t-1} + \alpha(m_t - p_{t-1}), \quad \text{with} \quad 0 < \alpha < 1.$$

Show that the autoregressive representation for price changes is

$$\phi(L)\Delta p_t = \varepsilon_t \quad \text{where} \quad \phi(L) = 1 - (1-\alpha)L \quad \text{and} \quad \varepsilon_t = \alpha w_t.$$

while verifying that

$$\phi(1)^{-2}\sigma_\varepsilon^2 = \sigma_w^2.$$

Solution

We are given that the fundamental value evolves as

$$m_t = m_{t-1} + w_t, \quad w_t \sim \text{i.i.d. } (0, \sigma_w^2),$$

and that the observed price adjusts partially toward the fundamental value

$$p_t = p_{t-1} + \alpha(m_t - p_{t-1}), \quad 0 < \alpha < 1.$$

Rewriting the price adjustment equation:

$$\begin{aligned} p_t &= (1-\alpha)p_{t-1} + \alpha m_t \\ \Rightarrow \Delta p_t &= p_t - p_{t-1} = -\alpha p_{t-1} + \alpha m_t = \alpha(m_t - p_{t-1}) \end{aligned}$$

Substitute for $m_t = m_{t-1} + w_t$:

$$\begin{aligned} \Delta p_t &= \alpha(m_{t-1} + w_t - p_{t-1}) \\ &= \alpha(m_{t-1} - p_{t-1}) + \alpha w_t \end{aligned}$$

From the previous period

$$\Delta p_{t-1} = \alpha(m_{t-1} - p_{t-2}) \Rightarrow m_{t-1} = \frac{1}{\alpha}\Delta p_{t-1} + p_{t-2}$$

Now substitute this into the equation for m_t:

$$m_t = m_{t-1} + w_t = \left(\frac{1}{\alpha}\Delta p_{t-1} + p_{t-2}\right) + w_t$$

Hence

$$\begin{aligned} \Delta p_t &= \alpha(m_t - p_{t-1}) \\ &= \alpha\left(\frac{1}{\alpha}\Delta p_{t-1} + p_{t-2} + w_t - p_{t-1}\right) \\ &= \Delta p_{t-1} + \alpha(p_{t-2} - p_{t-1}) + \alpha w_t \\ &= (1-\alpha)\Delta p_{t-1} + \alpha w_t \end{aligned}$$

Define

$$\varepsilon_t = \alpha w_t$$

Then we obtain the AR(1) representation:

$$\Delta p_t = (1-\alpha)\Delta p_{t-1} + \varepsilon_t$$

In lag operator notation:

$$(1 - (1 - \alpha)L)\Delta p_t = \varepsilon_t$$

For the variance, from the above, we know

$$\varepsilon_t = \alpha w_t \Rightarrow \sigma_\varepsilon^2 = \alpha^2 \sigma_w^2$$

Also

$$\phi(L) = 1 - (1 - \alpha)L \Rightarrow \phi(1) = 1 - (1 - \alpha) = \alpha$$

Now verify

$$\phi(1)^{-2}\sigma_\varepsilon^2 = \frac{1}{\alpha^2} \cdot \alpha^2 \sigma_w^2 = \sigma_w^2$$

2 Advanced Econometrics

2.1 Market Maker with Autocorrelated Trades

Suppose the trade direction $q_t \in \{-1, +1\}$ follows a Markov chain with continuation probability $\alpha \in \left(\frac{1}{2}, 1\right)$. Let the fundamental value evolve as:

$$m_t = m_{t-1} + \lambda q_t + u_t, \quad u_t \sim \mathcal{N}(0, \sigma_u^2)$$

and transaction prices are given by $p_t = m_t + cq_t$. The market maker knows the model but does not observe m_t; she observes p_t, p_{t-1}, and assumes a prior $m_{t-1} \sim \mathcal{N}(\mu_{t-1}, \sigma_{t-1}^2)$.

(a) Derive the Bayesian posterior belief of the market maker about m_t after observing p_t.

(b) Under what condition is the market maker's posterior variance minimized?

(c) How does the correlation in trade direction ($\alpha > 0.5$) bias the MM's inference of m_t?

(d) Can you derive a Kalman-like recursive filter for $\mathbb{E}[m_t | p_t, p_{t-1}]$?

2.2 Generalized Roll with Execution Delays

Let the price be set with one-period stale information:

$$p_t = m_{t-1} + cq_t$$

and

$$m_t = m_{t-1} + \lambda q_t + u_t, \quad q_t \sim \text{AR}(1) : q_t = \phi q_{t-1} + \varepsilon_t$$

(a) Derive the expression for Δp_t and compute its autocovariances up to lag 2.

(b) Show under what condition $\text{Cov}(\Delta p_t, \Delta p_{t-1}) > 0$, even though in the standard Roll model it is negative.

(c) Propose a method to de-bias the estimate of c when a researcher wrongly assumes iid q_t.

Solution

Using the setup provided, we can compute the autocovariance for succesive price changes to be given as

$$\gamma_0 = 2c^2 + \lambda^2 + \sigma_u^2$$

$$\gamma_1 = c(\lambda - c)$$

The condition under which this auto-covariance is positive may be interesting. If we have $\lambda > c$ meaning that infomational effects on price changes outweigh the bid-ask bounce then we may find positive auto-correlation in prices, i.e. higher returns are followed by even higher returns. On the other hand, if $\lambda < c$ then the model converges to the standard bid-ask bounce model where the spread can be estimated from the auto-covariances.

2.3 Optimal Estimation under Stale Prices

Suppose:

$$p_t = m_t + cq_t, \quad m_t = m_{t-1} + \lambda q_t + u_t, \quad \text{Corr}(q_t, q_{t-1}) = \rho$$

Only $\{p_t\}$ is observable.

(a) Show how estimating λ and c using autocovariances of Δp_t is biased if $\rho \neq 0$.

(b) Derive a consistent estimator for λ using third-order moments.

(c) Propose a GMM system using moment conditions involving Δp_t, Δp_{t-1}, and $\Delta^2 p_t$.

Solution

Under the given model assumptions, we calculate the value of γ_0 to be given as

$$\lambda^2 + \sigma_u^2 + 2c^2(1-\rho)$$

We can clearly notice that this now depends on the correlation of successive trade directions. If buys tend to follow buys, we find $Corr.(q_t, q_{t-1}) > 0$ and alternately for sells. Under these conditions, the value of γ_0 may be different (i.e. in case we assume them to be i.i.d which is wrong) as we don't take into consideration the serial dependence of successive trade directions.

2.4 Variance Decomposition with Informed Order Flow

Extend the model:

$$q_t = \gamma x_t + \eta_t, \quad \text{Cov}(x_t, m_t) > 0$$

(a) Decompose $\text{Var}(p_t)$ into components due to informed trading, noise trading, and inventory effects.

(b) Show how c and γ interact to amplify/dampen information incorporation.

(c) Suppose $x_t \sim AR(1)$. Derive the dynamic response of p_t to a shock in x_t.

(d) Derive a signal-to-noise ratio metric and relate it to optimal liquidity provision.

Solution

We now consider the case where the information content of prices is now contaminated by the effects of noise trading. Hence, the trade direction indicator q_t is now noisy and is partially observable. The model now assumes

$$m_t = m_{t-1} + \lambda(\gamma x_t + \eta_t) + u_t$$

$$p_t = m_t + c(\gamma x_t + \eta_t)$$

with $Cov(x_t, m_t) > 0$ indicating that trade direction has an effect on the fundamental value through the permanent information impact (something like Kyle's lambda). Extracting auto-covariances, we have

$$\gamma_0 = (\gamma^2 + \eta^2)((\lambda + c)^2 + c^2) + \sigma_u^2$$

$$\gamma_1 = -2c(\lambda + c)\gamma^2 \rho$$

This now sheds light on an important result. Since, we have γ^2, we may find that significant correlation with succesive trades is able to increase the autoc-covariance, however, it doesn't matter if buys follow buys or buy follow sells as this component has equal weightage in both cases. However what matters is the parameter ρ which indicates the effect of trade direction on the fundamental value though an informational content context. This means, in case of noise trading, we need to pay more attention to adverse selection risks as these may be hidden within the prices.

2.5 Invertibility and State-Space Inference

Given:

$$\Delta p_t = (\lambda + c)q_t - cq_{t-1} + u_t$$

and q_t is Markovian,

(a) Write the observation and state equations for a state-space model.

(b) Determine whether the system is invertible.

(c) Propose an algorithm to estimate the latent state q_t and m_t recursively.